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Some New Integral Inequalities via Variant of Pompeiu's Mean Value Theorem

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Abstract. The main of this paper is to establish an inequality providing some better bounds for integral mean by using a mean value theorem. Our results generalize the results of Ahmad et. al in [8].

1. Introduction

The inequality of Ostrowski [7] gives us an estimate for the deviation of the values of a smooth function from its mean value. More precisely, if $f : [a, b] \to \mathbb{R}$ is a differentiable function with bounded derivative, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}\,t \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \left\| f' \right\|_\infty$$

for every $x \in [a, b]$. Moreover the constant $1/4$ is the best possible.

For a differentiable function $f : [a, b] \to \mathbb{R}$, $a \cdot b > 0$, Dragomir has proved in [2], using Pompeiu's mean value theorem [5], the following Ostrowski type inequality:

$$\left| \frac{a+b}{2} . \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}\,t \right| \leq D(x) \left\| f - \ell f' \right\|_\infty$$

where $\ell(t) = t,\ t \in [a, b]$, and

$$D(x) = \frac{(b-a)}{|x|} \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right].$$

In [4], Pecaric and Ungar proved a general estimate with the p-norm, $1 < p < \infty$, which for $p = 1$ will give the Dragomir [2] result.

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In [8], for a twice differentiable function $f : [a, b] \to \mathbb{R},\ a{\cdot}b > 0$ Farooq et. al gave the following integral inequality:

$$\left| \frac{a+b}{2}\left(\frac{2f(x)}{3x} - \frac{f'(x)}{2}\right) + \frac{1}{3}\left(\frac{bf(b)-af(a)}{b-a}\right) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}\,t \right|$$
$$\leq \frac{(b-a)}{3\,|x|}\left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right]\left\|2f - 2\ell f' + \ell^2 f''\right\|_\infty$$

where $\ell(u) = u,\ u \in [a, b]$.

The interested reader is also referred to ([2]-[4], [6], [8], [9]) for integral inequalities by using Pompeiu's mean value theorem.

In this paper, we establish an general form with the p-norm, $1 \leq p \leq \infty$, which will give the Ahmad et. al result for $p = \infty$. Our results generalize the results of Ahmad et. al in [8].

2. Main Results

Before stating the main results, we will give the following lemma proved by Pecaric and Ungar in [4]:

Lemma 2.1. *For $\frac{1}{p} + \frac{1}{q} = 1,\ 1 \leq p, q \leq \infty$, and $0 < a \leq x \leq b$, denote*

$$A(x,q) := \left(\int_a^x \left(\int_t^x \frac{t^q\,\mathrm{d}\,u}{u^{2q}}\right)\mathrm{d}\,t\right)^{\frac{1}{q}} + \left(\int_x^b \left(\int_x^t \frac{t^q\,\mathrm{d}\,u}{u^{2q}}\right)\mathrm{d}\,t\right)^{\frac{1}{q}} \tag{1}$$

where for $p = 1$, i.e. $q = \infty$, the integrals are to be interpreted as the ∞-norms, i.e. as maxima of the function $(u, t) \mapsto \frac{1}{u^2}$ on the corresponding domains of integration. Then,

$$A(x,q) = \left(\frac{a^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - a^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{\frac{1}{q}}$$
$$+ \left(\frac{b^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - b^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{\frac{1}{q}},$$

for $1 < p, q < \infty,\ p, q \neq 2$;

$$A(x,2) = \frac{1}{3}\left[\left(\ln\left(\frac{x}{a}\right)^3 + \frac{a^3}{x^3} - 1\right)^{\frac{1}{2}} + \left(\ln\left(\frac{x}{b}\right)^3 + \frac{b^3}{x^3} - 1\right)^{\frac{1}{2}}\right] = \lim_{q\to 2} A(x,q);$$

$$A\left(x,\infty\right) = \frac{a^2+b^2}{2x} + x - a - b = \lim_{q\to\infty} A(x,q);$$

$$A\left(x,1\right) = \frac{1}{a} + \frac{b}{x^2} = \lim_{q\to 1} A(x,q).$$

To prove our theorems, we need the following lemma:

Lemma 2.2. *Let $f:[a,b]\to\mathbb{R}$ be continuous function on $[a,b]$ and twice order differentiable function on (a,b) with $0<a<b$. Then for any $t,x\in[a,b]$, we have*

$$
\begin{aligned}
&tf(x)-xf(t)+xt\frac{f'(t)-f'(x)}{2}= \\
&\qquad \frac{xt}{2}\int_x^t\left[2f(u)-2uf'(u)+u^2f''(u)\right]\frac{1}{u^2}\,\mathrm{d}\,u.
\end{aligned}
\tag{2}
$$

Proof. Define $\Psi:\left[\frac{1}{b},\frac{1}{a}\right]\to\mathbb{R}$ by $\Psi(t):=t^2f\left(\frac{1}{t}\right)$. The function Ψ is continuously differentiable on $\left(\frac{1}{b},\frac{1}{a}\right)$, and for all $x_1,x_2\in\left[\frac{1}{b},\frac{1}{a}\right]$, we get

$$
\Psi'(x_1)-\Psi'(x_2)=\int_{x_2}^{x_1}\Psi''(t)\,\mathrm{d}\,t
$$

$$
=\int_{x_2}^{x_1}\left[2f\left(\frac{1}{t}\right)-\frac{2}{t}f'\left(\frac{1}{t}\right)+\frac{1}{t^2}f''\left(\frac{1}{t}\right)\right]\mathrm{d}\,t.
$$

Using the change of the variable in last integrals with $u=\frac{1}{t}$, we get

$$
\Psi'(x_1)-\Psi'(x_2)=-\int_{\frac{1}{x_2}}^{\frac{1}{x_1}}\left[2f(u)-2uf'(u)+u^2f''(u)\right]\frac{1}{u^2}\,\mathrm{d}\,u.
\tag{3}
$$

Denote $x_1=\frac{1}{x}$ and $x_2=\frac{1}{t}$. Then for all $x,t\in[a,b]$ from (3), we have

$$
\begin{aligned}
&\frac{2}{x}f(x)-f'(x)-\frac{2}{t}f(t)+f'(t)= \\
&\qquad \int_x^t\left[2f(u)-2uf'(u)+u^2f'(u)\right]\frac{1}{u^2}\,\mathrm{d}\,u
\end{aligned}
$$

which gives (2) and completes the proof. □

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be continuous function on $[a,b]$ and twice order differentiable function on (a,b) with $0<a<b$. Then for $\frac{1}{p}+\frac{1}{q}=1$, with $1\le p,q\le\infty$, and all $x\in[a,b]$, we have*

$$
\begin{aligned}
&\left|\frac{a+b}{2}\left(\frac{2f(x)}{3x}-\frac{f'(x)}{3}\right)+\frac{1}{3}\left(\frac{bf(b)-af(a)}{b-a}\right)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}\,t\right| \\
&\quad\le\frac{(b-a)^{\frac{1}{p}-1}}{3}\left\|2f-2\ell f'+\ell^2f''\right\|_p A(x,q)
\end{aligned}
\tag{4}
$$

where $\ell(u)=u,\ u\in[a,b]$.

Proof. From Lemma 2.2, we have

$$tf(x)-xf(t)+xt\frac{f'(t)-f'(x)}{2} = \frac{xt}{2}\int_x^t \left[2f(u)-2uf'(u)+u^2f'(u)\right]\frac{1}{u^2}\,\mathrm{d}\,u. \tag{5}$$

Integrating with respect to t on $[a,b]$ and dividing by $\frac{3x}{2}$, we get

$$\frac{(b^2-a^2)}{2}\left(\frac{2f(x)}{3x}-\frac{f'(x)}{3}\right)+\frac{bf(b)-af(a)}{3}-\int_a^b f(t)\,\mathrm{d}\,t = \int_a^b \frac{t}{3}\left(\int_x^t \left[2f(u)-2uf'(u)+u^2f'(u)\right]\frac{1}{u^2}du\right)\mathrm{d}\,t$$

and therefore

$$\begin{aligned}&\left|\frac{(b^2-a^2)}{2}\left(\frac{2f(x)}{3x}-\frac{f'(x)}{3}\right)+\frac{bf(b)-af(a)}{3}-\int_a^b f(t)\,\mathrm{d}\,t\right| \\ &\le \int_a^b \left(\left|\int_x^t \left|\left[2f(u)-2uf'(u)+u^2f''(u)\right]\frac{t}{3u^2}\right|du\right|\right)\mathrm{d}\,t \\ &= \int_a^x \left(\left|\int_x^t \left|\left[2f(u)-2uf'(u)+u^2f''(u)\right]\frac{t}{3u^2}\right|du\right|\right)\mathrm{d}\,t \\ &\quad+\int_x^b \left(\left|\int_x^t \left|\left[2f(u)-2uf'(u)+u^2f''(u)\right]\frac{t}{3u^2}\right|du\right|\right)\mathrm{d}\,t.\end{aligned} \tag{6}$$

Firstly, we consider the case $1<p,q<\infty$. By using Hölder's inequality, the sum in the last line of (6) can be written

$$\left(\int_a^x\left(\int_t^x \left|2f(u)-2uf'(u)+u^2f'(u)\right|^p\mathrm{d}\,u\right)\mathrm{d}\,t\right)^{\frac{1}{p}}\left(\int_a^x\left(\int_t^x \frac{t^q\,\mathrm{d}\,u}{u^{2q}3^q}\right)\mathrm{d}\,t\right)^{\frac{1}{q}}$$
$$+\left(\int_x^b\left(\int_x^t \left|2f(u)-2uf'(u)+u^2f''(u)\right|^p\mathrm{d}\,u\right)\mathrm{d}\,t\right)^{\frac{1}{p}}\left(\int_x^b\left(\int_x^t \frac{t^q\,\mathrm{d}\,u}{u^{2q}3^q}\right)\mathrm{d}\,t\right)^{\frac{1}{q}}$$

(7)
$$\leq \frac{1}{3}\left(\int_a^b\left(\int_a^b\left|2f(u)-2uf'(u)+u^2f''(u)\right|^p \mathrm{d}\,u\right)\mathrm{d}\,t\right)^{\frac{1}{p}} \times\left[\left(\int_a^x\left(\int_t^x\frac{t^q\,\mathrm{d}\,u}{u^{2q}}\right)\mathrm{d}\,t\right)^{\frac{1}{q}}+\left(\int_x^b\left(\int_x^t\frac{t^q\,\mathrm{d}\,u}{u^{2q}}\right)\mathrm{d}\,t\right)^{\frac{1}{q}}\right].$$

The first factor in (7) is equal with

(8)
$$\left(\int_a^b\left(\int_a^b\left|2f(u)-2uf'(u)+u^2f''(u)\right|^p \mathrm{d}\,u\right)\mathrm{d}\,t\right)^{\frac{1}{p}} = (b-a)^{\frac{1}{p}}\left\|2f-2\ell f'+\ell^2 f''\right\|_p.$$

and, by Lemma 2.1, the second factor equals $A(x,q)$. Thus, putting (8) into (6) and dividing $b-a$ gives the required inequality (4). □

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be continuous function on $[a,b]$ and twice order differentiable function on (a,b) with $0<a<b$, and let $w:[a,b]\to\mathbb{R}$ be a nonnegative integrable function. Then for $\frac{1}{p}+\frac{1}{q}=1$ with $1\leq p,q\leq\infty$ any $t,x\in[a,b]$, we have*

(9)
$$\left|\left(\frac{2f(x)-xf'(x)}{2x}\right)\int_a^b tw(t)\,\mathrm{d}\,t-\int_a^b w(t)\,f(t)\,\mathrm{d}\,t+\frac{1}{2}\int_a^b tw(t)\,f't)\,\mathrm{d}\,t\right|$$
$$\leq (b-a)^{\frac{1}{p}}\left\|2f-2\ell f'+\ell^2 f''\right\|_p\left[\left(\int_a^x\frac{\left[x^{1-2q}t^q-t^{1-q}\right]w^q(t)}{(1-2q)}\mathrm{d}\,t\right)^{\frac{1}{q}} +\left(\int_x^b\frac{\left[t^{1-q}-x^{1-2q}t^q\right]w^q(t)}{(1-2q)}\mathrm{d}\,t\right)^{\frac{1}{q}}\right]$$

where $\ell(u)=u,\ u\in[a,b]$.

Proof. Multiplying (5) by $\frac{w(t)}{x}$ and integrating with respect to t on $[a,b]$, we have

$$\left(\frac{2f(x)-xf'(x)}{2x}\right)\int_a^b tw(t)\,\mathrm{d}\,t-\int_a^b w(t)\,f(t)dt+\frac{1}{2}\int_a^b tw(t)\,f'(t)\,\mathrm{d}\,t$$

$$= \frac{1}{2} \int_a^b t w(t) \left(\int_x^t \left[2uf(u) - 2uf'(u) + u^2 f''(u) \right] \frac{1}{u^2} \,\mathrm{d}\, u \right) \mathrm{d}\, t$$

and as in the proof of Theorem 2.1, we get

$$\left| \left(\frac{2f(x) - xf(x)}{2x} \right) \int_a^b t w(t) \,\mathrm{d}\, t - \int_a^b w(t) f(t) \,\mathrm{d}\, t + \frac{1}{2} \int_a^b t w(t) f'(t) \,\mathrm{d}\, t \right|$$

$$\leq \frac{1}{2} \int_a^b \left| \int_x^t \left| 2f(u) - 2uf'(u) + u^2 f''(u) \right| \frac{t w(t)}{u^2} \,\mathrm{d}\, u \right| \mathrm{d}\, t$$

$$= \int_a^x \left(\int_t^x \left| 2f(u) - 2uf'(u) + u^2 f''(u) \right| \frac{t w(t)}{u^2} \,\mathrm{d}\, u \right) \mathrm{d}\, t$$

$$+ \int_x^b \left(\int_x^t \left| 2f(u) - 2uf'(u) + u^2 f''(u) \right| \frac{t w(t)}{u^2} \,\mathrm{d}\, u \right) \mathrm{d}\, t$$

$$\leq \left[\int_a^x \int_c^y \left(\int_t^x \int_s^y \left| 2f(u) - 2uf'(u) + u^2 f''(u) \right|^p \mathrm{d}\, u \right) \mathrm{d}\, t \right]^{\frac{1}{p}}$$

$$\cdot \left[\int_a^x \left(\int_t^x \frac{t^q w^q(t) \,\mathrm{d}\, u}{u^{2q}} \right) \mathrm{d}\, t \right]^{\frac{1}{q}}$$

$$+ \left[\int_x^b \left(\int_x^t \left| 2f(u) - 2uf'(u) + u^2 f''(u) \right|^p \mathrm{d}\, u \right) \mathrm{d}\, t \right]^{\frac{1}{p}}$$

$$\cdot \left[\int_x^b \left(\int_x^t \frac{t^q w^q(t) \,\mathrm{d}\, u}{u^{2q}} \right) \mathrm{d}\, t \right]^{\frac{1}{q}}$$

$$\leq \left[\int_a^b \left(\int_a^b \left| 2f(u) - 2uf'(u) + u^2 f''(u) \right|^p \mathrm{d}\, u \right) \mathrm{d}\, t \right]^{\frac{1}{p}}$$

$$\times \left(\left[\int_a^x \left(\int_t^x \frac{t^q w^q(t) \,\mathrm{d}\, u}{u^{2q}} \right) \mathrm{d}\, t \right]^{\frac{1}{q}} + \left[\int_x^b \left(\int_x^t \frac{t^q w^q(t) \,\mathrm{d}\, u}{u^{2q}} \right) \mathrm{d}\, t \right]^{\frac{1}{q}} \right)$$

which gives (9). $\square$

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